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## Comodule algebras and integrable systems

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### Abstract

A method of constructing both classical and quantum completely integrable systems from (Jordan–Lie) comodule algebras is introduced. Several integrable models based on a  $so(2, 1)$  comodule algebra, two non-standard Schrödinger comodule algebras, the (classical and quantum)  $q$ -oscillator algebra and the reflection equation algebra are explicitly obtained.

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### 1. Introduction

In this paper, we present a generalization of the construction of integrable systems with coalgebra symmetry given in [1] (let us quote, for instance, [2–5] as different applications) by using the notion of comodule algebras. Essentially, the coalgebra approach [1] allowed the construction of integrable systems on the space

$$\overbrace{A \otimes A \otimes \cdots \otimes A}^N$$

starting from a Hamiltonian defined on a (either Poisson or non-commutative) Hopf algebra  $A$  with a number of Casimir operators/functions. As we shall see in the following, the comodule algebra approach will generalize this construction to systems defined on

$$\tilde{A} \otimes \overbrace{B \otimes \cdots \otimes B}^{N-1}$$

where the ‘one-body’ Hamiltonian is again defined on the algebra  $\tilde{A}$ . But now  $\tilde{A}$  need not be a Hopf algebra, but only a comodule algebra of another Hopf algebra  $B$ . We recall that the comodule algebras naturally appear, for instance, when the notion of covariance is implemented in the context of non-commutative geometry (if a given quantum space  $\mathcal{Q}$  is covariant under the action of a quantum group  $G_q$ , the algebra  $\mathcal{Q}$  is a  $G_q$ -comodule algebra [6–9]). We would like

to mention that the existence of a Poisson analogue of such comodule algebra generalization was already pointed out in [10].

In section 2, we introduce the general framework of Jordan–Lie algebras [11] as a useful tool in order to describe simultaneously the ‘algebraic integrability’ of both classical and quantum systems. Section 3 is devoted to the generalization of the formalism presented in [1]. The first extension is obtained by making use of homomorphisms of Jordan–Lie algebras with Casimir elements. In this context, the comodule algebra construction arises in a natural way when the previous homomorphism is just a coaction between a given comodule algebra  $A$  and a second Hopf algebra  $B$ .

In order to illustrate the formalism, several integrable systems are constructed in section 4. The first one is derived from a coaction  $\phi : so(2, 2) \rightarrow so(2, 2) \otimes so(2, 1)$ . Secondly, two new integrable deformations of the  $N$ -dimensional isotropic oscillator are constructed from two different (Poisson) Schrödinger comodule algebras. The  $q$ -oscillator algebra [12–14] is also shown to provide an example of an integrable system with  $su(2)_q$ -comodule algebra symmetry (this system was introduced for the first time in [14, 15]). Moreover, a classical  $q$ -oscillator is defined as a Poisson comodule algebra with respect to the Poisson  $su_q(2)$  algebra. From this, a classical version of the Kulish Hamiltonian can be constructed, as well as the classical analogue of the Jordan–Schwinger realization. Finally, it is shown how the reflection equation algebra [16] provides another interesting example of an  $N$ -dimensional integrable system related to quantum spaces and endowed with comodule algebra symmetry (see also [14]). In the concluding section we make a few comments and outline some further generalizations of the method presented here.

## 2. Jordan–Lie algebras and algebraic integrability

### 2.1. Jordan–Lie algebras

We tersely recall the algebraic structure of classical and quantum observables. A classical observable is a smooth function  $\mathcal{F} : \mathcal{P} \rightarrow \mathbb{R}$  where  $\mathcal{P}$  is a Poisson manifold. The space of classical observables is naturally equipped with two bilinear operations: the pointwise multiplication  $\cdot$

$$(f \cdot g)(x) := f(x)g(x) \quad x \in \mathcal{P}$$

and the Poisson bracket  $[\cdot, \cdot]$  given in local coordinates by

$$[f, g](x) := \sum_{i,j}^N \mathcal{P}_{ij}(x) \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} \quad x = (x_1, \dots, x_N)$$

where  $\mathcal{P}_{ij}$  is a Poisson tensor. The pointwise multiplication is a unital, associative and commutative product, while the Poisson bracket is a Lie product; the two structures are coupled by the Leibnitz rule (i.e. the Lie product is a derivation for the associative product).

On the other hand, a quantum observable is a self-adjoint operator on a given Hilbert space. The space of quantum observables is again equipped with two bilinear operations defined through the operator composition: the anticommutator

$$(a \cdot b) := ab + ba$$

that is a commutative but not associative unital product, and the Lie product

$$[a, b] := (ab - ba).$$

We note that both the ‘classical’ and the ‘quantum’ products obey the Jordan identity, namely:

$$(a \cdot b) \cdot a^2 = a \cdot (b \cdot a^2).$$

Hence both classical and quantum observables give rise to ‘Jordan–Lie algebras’ [11].

It is straightforward to generalize the definition of Jordan–Lie algebra by replacing the Jordan product  $\cdot$  with an associative but not necessarily commutative product  $\circ$ . This is the kind of Jordan–Lie algebra that we shall consider in this paper: while the classical product will be again given by pointwise multiplication, the quantum product will be just the operator (non-commutative) composition:

$$(a \circ b) := ab$$

**Definition 1.** A generalized Jordan–Lie algebra is a vector space  $\mathcal{A}$  equipped with two bilinear maps  $\circ$  and  $[\ , \ ]$  such that for all  $a, b, c \in \mathcal{A}$ :

$$\begin{aligned} a \circ b &\in \mathcal{A} & (a \circ b) \circ c &= a \circ (b \circ c) \\ [a, b] &\in \mathcal{A} & [a, b] &= -[b, a] \\ [a, [b, c]] + [c, [a, b]] + [b, [c, a]] &= 0 \\ [a, b \circ c] &= b \circ [a, c] + [a, b] \circ c. \end{aligned}$$

**Remark.** In the following, we will restrict considerations either to ‘classical Jordan–Lie’ algebras (CJL), where the associative product  $\circ$  is commutative, or to ‘quantum Jordan–Lie’ algebras (QJL), where the Lie product  $[\ , \ ]$  is defined through the associative product  $\circ$  as  $[a, b] = (a \circ b - b \circ a)$ .

## 2.2. Dynamics and algebraic integrability

From a purely algebraic point of view we can describe a Hamiltonian system as a pair  $(\mathcal{A}, H)$  where  $\mathcal{A}$  is a generalized (classical or quantum) Jordan–Lie algebra and  $H$  is a distinguished element of  $\mathcal{A}$  such that the time evolution of any other element  $x$  of  $\mathcal{A}$  is given by

$$\dot{x} = [x, H]. \tag{1}$$

Our aim is to construct classical and quantum integrable systems. Within the previous context, this will be achieved by constructing (classical or quantum) Jordan–Lie algebras that contain a ‘sufficiently large’ Abelian subalgebra

$$\mathcal{C} = \{x_i \in \mathcal{A} \ / \ [x_i, x_j] = 0\}$$

and by taking a given element of  $\mathcal{C}$  as the Hamiltonian of the system. Under such a Hamiltonian, (1) implies that all the elements of  $\mathcal{C}$  are constants of the motion in involution. In order to ensure complete integrability, the dimension of the Abelian subalgebra should be equal to the number of degrees of freedom of  $H$  under a certain (either symplectic or quantum-mechanical) realization. In addition, under such physical realizations, the commuting elements have to be functionally independent.

## 3. A constructive formalism

We remark that both CJL and QJL are closed under the tensor product, i.e. if  $A$  and  $B$  are CJL or QJL, the same holds for  $A \otimes B$  with the definitions:

$$\begin{aligned} (a \otimes b) \circ_{A \otimes B} (a' \otimes b') &= a \circ_A a' \otimes b \circ_B b' \\ [a \otimes b, a' \otimes b']_{A \otimes B} &= [a, a']_A \otimes b \circ_B b' + a' \circ_A a \otimes [b, b']_B. \end{aligned}$$

In the following, we will skip the subscripts labelling the spaces we are acting on.

Let us consider a set  $\{A_1, \dots, A_N\}$  of CJL or QJL; then  $A_1 \otimes \dots \otimes A_N$  is again endowed with a Jordan–Lie algebra structure. We will take this Jordan–Lie algebra as our algebra of observables. The following theorem yields an algorithmic procedure to obtain CJL or QJL with large Abelian subalgebras (a Poisson-map version of this construction was already mentioned in [10]). Its proof is obtained by a straightforward computation.

**Theorem 1.** *Let us assume that two (classical or quantum) Jordan–Lie algebras  $A_1, A_2$  are given such that  $A_1$  has a Casimir  $C$ , and there exists a linear map*

$$\phi : A_1 \rightarrow A_1 \otimes A_2$$

which is a Jordan–Lie algebra homomorphism:

$$\begin{aligned} [\phi(a), \phi(b)] &= \phi([a, b]) & \forall a, b \in A_1 \\ \phi(ab) &= \phi(a)\phi(b) & \forall a, b \in A_1. \end{aligned}$$

Let us define the following sequence of homomorphisms:

$$\phi^{(2)} \equiv \phi \quad \phi^{(i)} = (\phi^{(2)} \otimes \overbrace{id \otimes \dots \otimes id}^{i-2}) \circ \phi^{(i-1)} \quad i = 3, \dots, N$$

where clearly

$$\phi^{(i)} : A_1 \rightarrow A_1 \otimes \overbrace{A_2 \otimes \dots \otimes A_2}^{i-1}.$$

Then, the elements  $C^{(i)}$  given by

$$\begin{aligned} C^{(2)} &= \phi^{(2)}(C) \otimes \overbrace{1 \otimes \dots \otimes 1}^{N-2} \\ &\vdots \\ C^{(N-1)} &= \phi^{(N-1)}(C) \otimes 1 \\ C^{(N)} &= \phi^{(N)}(C) \end{aligned}$$

commute within  $A_1 \otimes \overbrace{A_2 \otimes \dots \otimes A_2}^{N-1}$ . Moreover,

$$[\phi^{(N)}(a), C^{(i)}] = 0 \quad \forall a \in A_1 \quad i = 2, \dots, N.$$

A relevant example of such a map  $\phi$  is given by a coaction mapping from a Jordan–Lie algebra to another Jordan–Lie Hopf algebra.

### 3.1. Integrable systems from comodule algebras

Let us recall the definition of a coaction of a Hopf algebra  $H$  on a vector space  $V$ :

**Definition 2.** *A (right) coaction of a Hopf algebra  $(H, \Delta)$  on a vector space  $V$  is a linear map  $\phi : V \rightarrow V \otimes H$  such that*

$$(\phi \otimes id) \circ \phi = (id \otimes \Delta) \circ \phi$$

i.e. if the following diagram is commutative:

$$\begin{array}{ccc}
 & V \otimes H & \\
 \phi \nearrow & & \searrow \phi \otimes id \\
 V & & V \otimes H \otimes H \\
 \phi \searrow & & \nearrow id \otimes \Delta \\
 & V \otimes H &
 \end{array}$$

If  $V$  is an algebra, we shall say that  $V$  is an  $H$ -comodule algebra if the coaction  $\phi$  is a homomorphism with respect to the product on the algebra

$$\phi(ab) = \phi(a)\phi(b) \quad \forall a, b \in V.$$

Moreover, if  $V$  has a Jordan–Lie structure and  $H$  is a Lie–Hopf algebra,  $V$  will also be a Jordan–Lie  $H$ -comodule algebra if

$$\phi([a, b]) = [\phi(a), \phi(b)] \quad \forall a, b \in V$$

So, given a (classical or quantum) Jordan–Lie–Hopf algebra with at least one Casimir  $C$ , if we find a coaction  $\phi$  of this Lie–Hopf algebra on a (classical or quantum) Jordan–Lie algebra, then we have all the ingredients to apply theorem 1.

Note that any Hopf algebra  $H$  is a  $H$ -comodule algebra with respect to itself, since the coaction map is given by the coproduct map between  $H$  and  $H \otimes H$ . Therefore, the coalgebra symmetry approach is just a particular case of theorem 1. Many examples of integrable systems with this type of coalgebra symmetry have already been found (see, for instance, [1–5]).

## 4. Examples

### 4.1. Classical systems with an $so(2, 1)$ comodule algebra symmetry

We consider the Poisson  $so(2, 2)$  Lie algebra with generators  $\{J_3, J_{\pm}, N_3, N_{\pm}\}$  and Poisson brackets:

$$\begin{aligned}
 \{J_3, J_{\pm}\} &= \{N_3, N_{\pm}\} = \pm 2J_{\pm} \\
 \{J_3, N_{\pm}\} &= \{N_3, J_{\pm}\} = \pm 2N_{\pm} \\
 \{J_+, J_-\} &= \{N_+, N_-\} = J_3 \\
 \{J_{\pm}, N_{\mp}\} &= \pm N_3 \quad \{J_m, N_m\} = 0 \quad m = +, -, 3.
 \end{aligned} \tag{2}$$

The two (second-order) Casimir functions for this algebra are

$$C_1 = \frac{1}{2}J_3^2 + \frac{1}{2}N_3^2 + J_+J_- + J_-J_+ + N_+N_- + N_-N_+ \tag{3}$$

$$C_2 = \frac{1}{2}J_3N_3 + J_+N_- + J_-N_+. \tag{4}$$

The algebra  $so(2, 2)$  can be seen as an  $so(2, 1)$ -comodule algebra through a coaction

$$\phi^{(2)} : so(2, 2) \rightarrow so(2, 2) \otimes so(2, 1)$$

given by the map

$$\begin{aligned}
 \phi^{(2)}(J_3) &= J_3 \otimes 1 + 1 \otimes Y_3 & \phi^{(2)}(N_3) &= N_3 \otimes 1 + 1 \otimes Y_3 \\
 \phi^{(2)}(J_+) &= J_+ \otimes 1 + 1 \otimes Y_+ & \phi^{(2)}(N_+) &= N_+ \otimes 1 + 1 \otimes Y_+ \\
 \phi^{(2)}(J_-) &= J_- \otimes 1 + 1 \otimes Y_- & \phi^{(2)}(N_-) &= N_- \otimes 1 + 1 \otimes Y_-
 \end{aligned}$$

where the Lie brackets in  $so(2, 1)$  are

$$\{Y_3, Y_{\pm}\} = \pm 2Y_{\pm} \quad \{Y_+, Y_-\} = Y_3$$

and the coproduct in  $so(2, 1)$  is the primitive one

$$\Delta(X) = X \otimes 1 + 1 \otimes X \quad X = Y_{\pm}, Y_3.$$

It is immediate to define the chain of homomorphisms

$$\phi^{(i)} = (\phi^{(2)} \otimes \overbrace{id \otimes \cdots \otimes id}^{i-2}) \circ \phi^{(i-1)} \quad i = 3, \dots, N.$$

As a consequence, the set of  $2N$  operators

$$C_1^{(i)} = \phi^{(i)}(C_1) \otimes \overbrace{1 \otimes 1 \otimes \cdots \otimes 1}^{N-i} \quad (5)$$

$$C_2^{(i)} = \phi^{(i)}(C_2) \otimes \overbrace{1 \otimes 1 \otimes \cdots \otimes 1}^{N-i} \quad (6)$$

are in involution with all the elements of the subalgebra  $\phi^{(N)}(so(2, 2))$  within  $so(2, 2) \otimes so(2, 1)^{\otimes(N-1)}$ .

In order to get classical mechanical systems we can consider the symplectic realization  $D$  of  $so(2, 2)$  given by

$$\begin{aligned} D(J_3) &= p_1 q_1 + p_2 q_2 \\ D(J_+) &= \frac{1}{2} (p_1^2 + p_2^2) + \frac{a_{12}}{(q_1 + q_2)^2} + \frac{b_{12}}{(q_1 - q_2)^2} \\ D(J_-) &= -\frac{1}{2} (q_1^2 + q_2^2) \\ D(N_3) &= (p_1 q_2 + p_2 q_1) \\ D(N_+) &= p_1 p_2 + \frac{a_{12}}{(q_1 + q_2)^2} - \frac{b_{12}}{(q_1 - q_2)^2} \\ D(N_-) &= -q_1 q_2. \end{aligned} \quad (7)$$

It is readily checked that the functions (7) close the  $so(2, 2)$  Poisson algebra. This symplectic realization is characterized by the Casimirs, whose values are  $D(C_1) = -(a_{12} + b_{12})$  and  $D(C_2) = -(a_{12} - b_{12})/2$ . (Note that  $D(J_+)$  is just the two-body rational Calogero–Moser Hamiltonian.)

Secondly, we consider the symplectic realization  $S$  of  $so(2, 1)$  defined by

$$S(Y_3) = p_3 q_3 \quad S(Y_+) = \frac{p_3^2}{2} + \frac{c_3}{q_3^2} \quad S(Y_-) = -\frac{q_3^2}{2}.$$

By using these realizations, the image of the generators under the coaction is

$$\begin{aligned} \phi^{(2)}(J_3) &= p_1 q_1 + p_2 q_2 + p_3 q_3 \\ \phi^{(2)}(J_+) &= \frac{1}{2} (p_1^2 + p_2^2) + \frac{a_{12}}{(q_1 + q_2)^2} + \frac{b_{12}}{(q_1 - q_2)^2} + \frac{p_3^2}{2} + \frac{c_3}{q_3^2} \\ \phi^{(2)}(J_-) &= -\frac{1}{2} (q_1^2 + q_2^2) - \frac{q_3^2}{2} \\ \phi^{(2)}(N_3) &= p_1 q_2 + p_2 q_1 + p_3 q_3 \\ \phi^{(2)}(N_+) &= p_1 p_2 + \frac{a_{12}}{(q_1 + q_2)^2} - \frac{b_{12}}{(q_1 - q_2)^2} + \frac{p_3^2}{2} + \frac{c_3}{q_3^2} \\ \phi^{(2)}(N_-) &= -q_1 q_2 - \frac{q_3^2}{2} \end{aligned} \quad (8)$$

and the image of the Casimir functions  $C_1$  and  $C_2$  is no longer a constant:

$$\begin{aligned} C_1^{(2)} &:= (D \otimes S)(C_1) = -(a_{12} + b_{12} + 2c_3) + p_3 q_3 (p_1 + p_2)(q_1 + q_2) \\ &\quad - \left( \frac{1}{2} p_3^2 + \frac{c_3}{q_3^2} \right) (q_1 + q_2)^2 - q_3^2 \left[ \frac{1}{2} (p_1 + p_2)^2 + \frac{2a_{12}}{(q_1 + q_2)^2} \right] \\ C_2^{(2)} &:= (D \otimes S)(C_2) = \frac{1}{2} C_1^{(2)} + b_{12} \end{aligned} \quad (9)$$

meaning that both Casimirs give rise to functionally dependent images. Therefore, the function  $C_1^{(2)}$  and the image of two Poisson-commuting generators of  $so(2, 2)$  (for instance, we could take  $\phi(J_+)$  and  $\phi(N_+)$ ) give us a complete family of Poisson-commuting observables and allow us to construct a three-body completely integrable Hamiltonian system.

The  $N$ th image would be given by

$$\begin{aligned} \phi^{(N)}(J_3) &= p_1 q_1 + p_2 q_2 + \sum_{k=3}^{N+1} p_k q_k \\ \phi^{(N)}(J_+) &= \frac{1}{2} (p_1^2 + p_2^2) + \frac{a_{12}}{(q_1 + q_2)^2} + \frac{b_{12}}{(q_1 - q_2)^2} + \sum_{k=3}^{N+1} \left( \frac{p_k^2}{2} + \frac{c_k}{q_k^2} \right) \\ \phi^{(N)}(J_-) &= -\frac{1}{2} (q_1^2 + q_2^2) - \sum_{k=3}^{N+1} \frac{q_k^2}{2} \\ \phi^{(N)}(N_3) &= p_1 q_2 + p_2 q_1 + \sum_{k=3}^{N+1} p_k q_k \\ \phi^{(N)}(N_+) &= p_1 p_2 + \frac{a_{12}}{(q_1 + q_2)^2} - \frac{b_{12}}{(q_1 - q_2)^2} + \sum_{k=3}^{N+1} \left( \frac{p_k^2}{2} + \frac{c_k}{q_k^2} \right) \\ \phi^{(N)}(N_-) &= -q_1 q_2 - \sum_{k=3}^{N+1} \frac{q_k^2}{2} \end{aligned} \quad (10)$$

where we have used the notation

$$p_i = \overbrace{1 \otimes \cdots \otimes 1}^{i-1} \otimes p \otimes \overbrace{1 \otimes \cdots \otimes 1}^{N-i} \quad q_i = \overbrace{1 \otimes \cdots \otimes 1}^{i-1} \otimes q \otimes \overbrace{1 \otimes \cdots \otimes 1}^{N-i}.$$

A complete family of Poisson-commuting observables in the  $(N + 1)$ -body case is provided by the symplectic realizations of the images of the Casimirs

$$\begin{aligned} C_1^{(M)} &= - \left( a_{12} + b_{12} + 2 \sum_{k=3}^{M+1} c_k \right) + \sum_{k=3}^{M+1} p_k q_k (p_1 + p_2)(q_1 + q_2) \\ &\quad - \sum_{j=3}^{M+1} \sum_{k>j}^{M+1} (q_j p_k - p_k q_j)^2 - \sum_{k=3}^{M+1} \left( \frac{1}{2} p_k^2 + \frac{c_k}{q_k^2} \right) (q_1 + q_2)^2 \\ &\quad - 2 \sum_{j,k=3}^{M+1} q_j^2 \frac{c_k}{q_k^2} - \sum_{k=3}^{M+1} q_k^2 \left[ \frac{1}{2} (p_1 + p_2)^2 + \frac{2a_{12}}{(q_1 + q_2)^2} \right] \quad M = 2, \dots, N \end{aligned}$$

together with two commuting elements (for instance,  $\phi^{(N)}(J_+)$  and  $\phi^{(N)}(N_+)$ ).



#### 4.2. Poisson–Schrödinger comodule algebras

Certain subalgebras of Hopf algebras provide examples of comodule algebras which are useful in order to construct new integrable systems. In particular, let  $(B, \Delta_B)$  be a Hopf algebra and let  $A \subset B$  be a subalgebra of  $B$  (as an algebra, but not as a Hopf algebra: this means that the coproduct  $\Delta_B$  of the elements of  $A$  contains some elements which do not belong to  $A \otimes A$ ). In this case it is straightforward to prove that, if there exists a basis in  $B$  such that the coproduct  $\Delta_B$  of the elements of  $A$  is of the form

$$\Delta_B(X) = \sum_{\alpha} X_{\alpha} \otimes Y_{\alpha} \quad \forall X \in A \quad \text{where} \quad X_{\alpha} \in A \quad \text{and} \quad Y_{\alpha} \in B \quad (11)$$

then  $A$  is a  $B$ -comodule algebra through the coaction  $\phi : A \rightarrow A \otimes B$  given by

$$\phi(X) := \Delta_B(X) \quad \forall X \in A. \quad (12)$$

In the non-commutative case, this situation is quite usual within the quantum algebra approach to discrete symmetries (see, for instance, [17, 18]). In the following, we will consider as  $B$  the Poisson analogues of two non-standard deformations of the Schrödinger algebra introduced in [17]. The associated (‘deformed’) coactions on a  $gl(2)$  subalgebra (that plays the role of  $A$ ) will give rise to two new integrable deformations of the isotropic oscillator in  $N$ -dimensions.

*4.2.1. The ‘space-type’ comodule Schrödinger system.* Let us consider the following Schrödinger Poisson–Hopf algebra  $\mathcal{S}_{\sigma} \equiv B$ :

$$\begin{aligned} \{\mathcal{D}, \mathcal{P}\} &= -\mathcal{P} & \{\mathcal{D}, \mathcal{K}\} &= \mathcal{K} & \{\mathcal{K}, \mathcal{P}\} &= \mathcal{M} & \{\mathcal{M}, \cdot\} &= 0 \\ \{\mathcal{D}, \mathcal{H}\} &= -2\mathcal{H} & \{\mathcal{D}, \mathcal{C}\} &= 2\mathcal{C} & \{\mathcal{H}, \mathcal{C}\} &= \mathcal{D} & \{\mathcal{H}, \mathcal{P}\} &= 0 \\ \{\mathcal{P}, \mathcal{C}\} &= -\mathcal{K} & \{\mathcal{K}, \mathcal{H}\} &= \mathcal{P} & \{\mathcal{K}, \mathcal{C}\} &= 0 \end{aligned} \quad (13)$$

and the deformed coproducts compatible with these Poisson brackets are found to be

$$\begin{aligned} \Delta(\mathcal{M}) &= 1 \otimes \mathcal{M} + \mathcal{M} \otimes 1 \\ \Delta(\mathcal{H}) &= 1 \otimes \mathcal{H} + \mathcal{H} \otimes (1 + \sigma\mathcal{P})^2 \\ \Delta(\mathcal{D}) &= 1 \otimes \mathcal{D} + \mathcal{D} \otimes \frac{1}{1 + \sigma\mathcal{P}} - \frac{1}{2} \mathcal{M} \otimes \frac{\sigma\mathcal{P}}{1 + \sigma\mathcal{P}} \\ \Delta(\mathcal{C}) &= 1 \otimes \mathcal{C} + \mathcal{C} \otimes \frac{1}{(1 + \sigma\mathcal{P})^2} + \sigma\mathcal{D}' \otimes \frac{1}{1 + \sigma\mathcal{P}} \mathcal{K} + \frac{\sigma^2}{2} (\mathcal{D}')^2 \otimes \frac{\mathcal{M}}{(1 + \sigma\mathcal{P})^2} \\ \Delta(\mathcal{P}) &= 1 \otimes \mathcal{P} + \mathcal{P} \otimes 1 + \sigma\mathcal{P} \otimes \mathcal{P} \\ \Delta(\mathcal{K}) &= 1 \otimes \mathcal{K} + \mathcal{K} \otimes \frac{1}{1 + \sigma\mathcal{P}} + \sigma\mathcal{D}' \otimes \frac{\mathcal{M}}{1 + \sigma\mathcal{P}} \end{aligned} \quad (14)$$

where  $\mathcal{D}' = \mathcal{D} + \frac{1}{2}\mathcal{M}$ . This algebra is the Poisson analogue of the ‘discrete space’ non-standard quantum Schrödinger algebra of [17], where  $\sigma$  is the deformation parameter (if  $\sigma \rightarrow 0$ , we recover the primitive coproduct  $\Delta(X) = X \otimes 1 + 1 \otimes X$ ).

If we consider the Poisson- $gl(2)$  subalgebra generated by  $\{\mathcal{M}, \mathcal{H}, \mathcal{D}, \mathcal{C}\}$  as our subalgebra  $A$ , it is straightforward to check that the Poisson- $gl(2)$  subalgebra fulfils condition (11). As a consequence,  $gl(2)$  is a (Poisson)- $\mathcal{S}_{\sigma}$  comodule algebra and (14) can be used to define a (right) coaction map  $\phi^{(2)} : gl(2) \rightarrow gl(2) \otimes \mathcal{S}_{\sigma}$  in the form

$$\phi^{(2)}(X) := \Delta(X) \quad X \in \{\mathcal{M}, \mathcal{H}, \mathcal{D}, \mathcal{C}\}. \quad (15)$$

A symplectic realization of  $\mathcal{S}_{\sigma}$  in terms of a pair of canonical coordinates is given by

$$\begin{aligned} S(\mathcal{C}) &= \frac{q_1^2}{2} & S(\mathcal{H}) &= \frac{p_1^2}{2} & S(\mathcal{D}) &= -p_1 q_1 \\ S(\mathcal{M}) &= \lambda_1^2 & S(\mathcal{K}) &= \lambda_1 q_1 & S(\mathcal{P}) &= \lambda_1 p_1 \end{aligned} \quad (16)$$

Under this one-particle symplectic realization, the  $gl(2)$  Casimir function

$$C_A = \frac{1}{4}D^2 - \mathcal{H}\mathcal{C} \quad (17)$$

vanishes  $S(C_A) = 0$  (note that  $\mathcal{M}$  is also a trivial central element).

If we take as the Hamiltonian on  $gl(2)$  the function

$$H = \mathcal{H} + \mathcal{C} \quad (18)$$

the symplectic realization  $S$  will give us the one-dimensional harmonic oscillator:

$$H^{(1)} := S(H) = S(\mathcal{H}) + S(\mathcal{C}) = \frac{p_1^2}{2} + \frac{q_1^2}{2}.$$

Now, by using the coaction map (15) we can define a Hamiltonian function on  $gl(2) \otimes \mathcal{S}_\sigma$ :

$$\phi^{(2)}(H) = \phi^{(2)}(\mathcal{H}) + \phi^{(2)}(\mathcal{C}) = \Delta(\mathcal{H}) + \Delta(\mathcal{C}).$$

This Hamiltonian can be expressed in terms of canonical coordinates by taking the symplectic realization  $S \otimes S$  of (14). It reads

$$\begin{aligned} H_\sigma^{(2)} = (S \otimes S)(\phi^{(2)}(H)) &= (S \otimes S)(\Delta(\mathcal{H}) + \Delta(\mathcal{C})) = \frac{1}{2}(p_1^2 + p_2^2) + \frac{q_2^2}{2} + \frac{q_1^2}{2(1 + \sigma\lambda_2 p_2)^2} \\ &+ \sigma\lambda_2 \left( p_1^2 p_2 + \frac{q_2(\lambda_1^2 - 2q_1 p_1)}{2(1 + \sigma\lambda_2 p_2)} \right) + \sigma^2 \lambda_2^2 \left( \frac{1}{2} p_1^2 p_2^2 + \frac{(\lambda_1^2 - 2q_1 p_1)^2}{8(1 + \sigma\lambda_2 p_2)^2} \right). \end{aligned} \quad (19)$$

This Hamiltonian is just an integrable deformation of the two-dimensional isotropic oscillator, since  $\lim_{\sigma \rightarrow 0} H^{(2)} = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}(q_1^2 + q_2^2)$ . The integral of motion is obtained as the coaction of the  $gl(2)$  Casimir (17):

$$\begin{aligned} C_\sigma^{(2)} &= (S \otimes S)(\phi^{(2)}(C_A)) = (S \otimes S) \left( \frac{1}{4} \Delta(D)^2 - \Delta(\mathcal{H}) \right) \\ &= - \frac{\{2(p_2 q_1 - p_1 q_2) + \sigma p_1(2p_1 q_1 - 4p_2 q_2 - \lambda_1^2) - \sigma^2 \lambda_2^2 p_1 p_2(-2p_1 q_1 + 2p_2 q_2 + \lambda_1^2)\}^2}{16(1 + \sigma\lambda_2 p_2)^2} \end{aligned} \quad (20)$$

As expected, the limit  $\sigma \rightarrow 0$  of (23) is just  $-(p_2 q_1 - p_1 q_2)^2/4$ .

By following theorem 1, further iterations of the coaction map would provide the corresponding integrable deformation of the isotropic oscillator in an arbitrary dimension.

**4.2.2. The ‘time-type’ comodule Schrödinger system.** Another integrable deformation of the isotropic oscillator can be obtained through the Poisson version of the ‘discrete space’ quantum Schrödinger algebra  $\mathcal{S}_\tau$  [17]. In this case, the deformed coproducts compatible with the Poisson-Schrödinger algebra (13) are

$$\begin{aligned} \Delta(\mathcal{M}) &= 1 \otimes \mathcal{M} + \mathcal{M} \otimes 1 \\ \Delta(\mathcal{H}) &= 1 \otimes \mathcal{H} + \mathcal{H} \otimes 1 + \tau \mathcal{H} \otimes \mathcal{H} \\ \Delta(\mathcal{D}) &= 1 \otimes \mathcal{D} + \mathcal{D} \otimes \frac{1}{1 + \tau \mathcal{H}} - \frac{1}{2} \mathcal{M} \otimes \frac{\tau \mathcal{H}}{1 + \tau \mathcal{H}} \\ \Delta(\mathcal{C}) &= 1 \otimes \mathcal{C} + \mathcal{C} \otimes \frac{1}{1 + \tau \mathcal{H}} - \frac{\tau}{2} \mathcal{D}' \otimes \frac{1}{1 + \tau \mathcal{H}} \mathcal{D} + \frac{\tau^2}{4} (\mathcal{D}')^2 \otimes \frac{\mathcal{H}}{(1 + \tau \mathcal{H})^2} \\ \Delta(\mathcal{P}) &= 1 \otimes \mathcal{P} + \mathcal{P} \otimes (1 + \tau \mathcal{H})^{1/2} \\ \Delta(\mathcal{K}) &= 1 \otimes \mathcal{K} + \mathcal{K} \otimes \frac{1}{(1 + \tau \mathcal{H})^{1/2}} + \frac{\tau}{2} \mathcal{D}' \otimes \frac{\mathcal{P}}{1 + \tau \mathcal{H}}. \end{aligned} \quad (21)$$

Once again, we can consider  $gl(2)$  as the subalgebra  $A$ , since the coproduct (21) of the generators  $\{\mathcal{M}, \mathcal{H}, \mathcal{D}, \mathcal{C}\}$  defines a coaction  $\phi^{(2)} : gl(2) \rightarrow gl(2) \otimes \mathcal{S}_\tau$ . By taking as the Hamiltonian on  $gl(2)$  the same function  $H = \mathcal{H} + \mathcal{C}$  and by considering the symplectic realization (16) we obtain

$$H_\tau^{(2)} = (S \otimes S)(\phi^{(2)}(H)) = (S \otimes S)(\Delta(\mathcal{H}) + \Delta(\mathcal{C})) = \frac{1}{2}(p_1^2 + p_2^2) + \frac{q_2^2}{2} + \frac{q_1^2}{2 + \tau p_2^2} \\ + \tau \left( \frac{1}{4} p_1^2 p_2^2 + \frac{p_2 q_2 (\lambda_1^2 - 2q_1 p_1)}{2(2 + \tau p_2^2)} \right) + \tau^2 \left( \frac{p_2 (\lambda_1^2 - 2q_1 p_1)}{8(2 + \tau p_2^2)} \right)^2. \quad (22)$$

In this case the integral of the motion reads

$$C_\tau^{(2)} = (S \otimes S)(\phi^{(2)}(C_A)) = (S \otimes S) \left( \frac{1}{4} \Delta(\mathcal{D})^2 - \Delta(\mathcal{H}) \right) \\ = - \frac{\{4(p_1 q_2 - p_2 q_1) - 2\tau p_1^2 p_2 q_1 + \tau p_1 p_2 (2p_2 q_2 + \lambda_1^2)\}^2}{32(2 + \tau p_2^2)}. \quad (23)$$

As in the previous case, the procedure can be extended to arbitrary dimensions. However, a compact and explicit expression for the  $N$ th coaction  $\phi^{(N)}$  is still lacking.

#### 4.3. $q$ -oscillator systems

Let us now consider as our vector space  $V$  the ‘ $q$ -oscillator algebra’  $A_q$  defined by the commutation relations [12–14]:

$$[N, A] = -A \quad [N, A^+] = A^+ \quad [A, A^+] = q^{-2N}$$

and as our coalgebra  $H$  we shall take the quantum algebra  $su_q(2)$  [19] defined by the commutation rules:

$$[J, X_\pm] = X_\pm \quad [X_+, X_-] = \frac{q^{2J} - q^{-2J}}{q - q^{-1}} = [2J]_q$$

and Casimir operator

$$L_q = [J]_q [J - 1]_q + X_+ X_-$$

where  $[j]_q [j + 1]_q$  is the eigenvalue of  $L_q$  in the  $(2j + 1)$ -dimensional irreducible representation of  $su_q(2)$ . The coaction  $\phi^{(2)}$  is defined by [14]

$$\phi^{(2)}(N) = N \otimes 1 + 1 \otimes J \\ \phi^{(2)}(A) = A \otimes q^J + (q - q^{-1})^{\frac{1}{2}} q^{-N} \otimes X_- \\ \phi^{(2)}(A^+) = A^+ \otimes q^J + (q - q^{-1})^{\frac{1}{2}} q^{-N} \otimes X_+.$$

It is easy to check that this coaction turns  $A_q$  into a (Jordan–Lie)  $su_q(2)$  comodule algebra. On the other hand,  $A_q$  is equipped with the Casimir

$$C_q = A^+ A - \frac{q^{-2N} - 1}{q^{-2} - 1}$$

which is not invariant under the coaction  $\phi^{(2)}$ .

4.3.1. *Quantum Hamiltonian.* If we take as Hamiltonian [14, 15]

$$H = A^+A$$

its image under the first coaction will be  $H^{(2)} = \phi^{(2)}(H) = \phi^{(2)}(A^+)\phi^{(2)}(A)$  and reads

$$H^{(2)} = A^+Aq^{2J} + (q - q^{-1})q^{-2N}X_+X_- + (q - q^{-1})^{\frac{1}{2}}q^{-N+J}\{q^{-1}AX_+ + qA^+X_-\}.$$

The coaction of the Casimir  $\phi^{(2)}(C_q)$  yields the constant of the motion  $C^{(2)}$ :

$$\phi^{(2)}(C_q) = \phi^{(2)}\left(A^+A - \frac{q^{-2N} - 1}{q^{-2} - 1}\right) = \phi^{(2)}(H) - \phi^{(2)}\left(\frac{q^{-2N} - 1}{q^{-2} - 1}\right)$$

namely

$$C^{(2)} = H^{(2)} - \frac{q^{-2(N+J)} - 1}{q^{-2} - 1}.$$

Since  $C^{(2)}$  is a constant of the motion, the conservation of  $(N + J)$ —the total number of excitations—follows.

In fact, the coaction  $\phi^{(2)}$  can be thought of as a homomorphism between  $A_q$  and  $A_q \otimes A_q \otimes A_q$ , since  $su_q(2)$  can be rewritten in terms of two  $q$ -oscillators through the  $q$ -analogue of the Jordan–Schwinger transformation [20, 21]:

$$J = \frac{1}{2}(N_1 - N_2) \quad X_+ = B^+q^{N_1/2}q^{N_2/2}C \quad X_- = q^{N_1/2}BC^+q^{N_2/2} \quad (24)$$

where  $\{N_1, B, B^+\}$  and  $\{N_2, C, C^+\}$  are two copies of the  $C_q = 0$  irreducible representation of the algebra  $A_q$  (note that the representation theory of the  $q$ -oscillator is much more complex than the oscillator one; this should be taken into account because different models could be obtained for each class of representations—see, for instance, [22]). In terms of this Jordan–Schwinger realization, the  $H^{(2)}$  Hamiltonian is rewritten as

$$H^{(2)} = A^+Aq^{N_1-N_2} + (q - q^{-1})q^{-2N} \left( [j]_q[j+1]_q - \left[\frac{1}{2}(N_1 - N_2)\right]_q \left[\frac{1}{2}(N_1 - N_2 - 2)\right]_q \right) \\ + (q - q^{-1})^{\frac{1}{2}}q^{-\frac{1}{2}(2N-3N_1+N_2+1)}\{q^{-1}AB^+C + qA^+BC^+\}.$$

We thus have a  $q$ -deformed three-wave interaction Hamiltonian that preserves the number operator  $N + N_1 - N_2$  and whose limit  $q \rightarrow 1$  is just  $A^+A$ .

The  $k$ -dimensional model is obtained through the  $k$ th iteration of the coaction map. In particular, the  $k$ -dimensional Hamiltonian reads

$$H^{(k)} = \phi^{(k)}(A^+A) = (A^+A)\Delta^{(k-1)}(q^{2J}) + (q - q^{-1})q^{-2N}\Delta^{(k-1)}(X_+X_-) \\ + (q - q^{-1})^{\frac{1}{2}}\{(q^{-N}A)\Delta^{(k-1)}(X_+q^J) + (A^+q^{-N})\Delta^{(k-1)}(q^JX_-)\}$$

where  $\Delta^{(k-1)}$  is the  $(k-1)$ th coproduct in  $su_q(2)$ . This Hamiltonian is completely integrable, since it commutes with the  $m$ th-order Casimirs, which are obtained as the corresponding images under the  $m$ th coaction:

$$C^{(m)} = \phi^{(m)}(C_q) = H^{(m)} - \phi^{(m)}\left(\frac{q^{-2(N)} - 1}{q^{-2} - 1}\right) = H^{(m)} - \left(\frac{q^{-2\phi^{(m)}(N)} - 1}{q^{-2} - 1}\right).$$

By construction, all these integrals are in involution

$$[C^{(l)}, C^{(p)}] = 0 \quad 2 \leq l < p \leq k.$$

In particular, the commutation rule  $[H^{(k)}, C^{(k)}] = 0$  implies that the  $\phi^{(k)}(N)$  operator commutes with the Hamiltonian. Finally, note that through the Jordan–Wigner transformation (24) the  $k$ th-order Hamiltonian can be seen as a system of  $(2k + 1)$  interacting  $q$ -oscillators.

4.3.2. *Classical model.* The previous model can be translated into classical mechanical terms by an appropriate definition of the ‘ $q$ -oscillator Poisson algebra’ that (to our knowledge) is a new one. The classical Poisson algebra  $A_q$  is defined by

$$\{N, A\} = -A \quad \{N, A^+\} = A^+ \quad \{A, A^+\} = e^{-2zN} \quad (25)$$

and as our coalgebra  $H$  we will take the  $su_q(2)$  Poisson coalgebra given by

$$\{J, X_{\pm}\} = X_{\pm} \quad \{X_+, X_-\} = \frac{e^{2zJ} - e^{-2zJ}}{2z}$$

The coaction  $\phi^{(2)}$  compatible with (25) is

$$\begin{aligned} \phi^{(2)}(N) &= N \otimes 1 + 1 \otimes J \\ \phi^{(2)}(A) &= A \otimes e^{zJ} + \sqrt{2z} e^{-zN} \otimes X_- \\ \phi^{(2)}(A^+) &= A^+ \otimes e^{zJ} + \sqrt{2z} e^{-zN} \otimes X_+. \end{aligned}$$

Since  $A_q$  is equipped with the Casimir function,

$$C_q = A^+A - \frac{1 - e^{-2zN}}{2z}$$

a  $k$ -dimensional Abelian subalgebra inside  $\phi^{(k)}(A_q)$  can be constructed through the approach described in the previous section. The results are similar to the ones for the ‘quantum’  $q$ -oscillator algebra, and the Jordan–Schwinger transformation is also valid in the Poisson case (with  $q = e^z$ ). Let us also mention that a symplectic realization of the Poisson algebra (25) corresponding to the value  $C_q = 0$  is given by the relations

$$N = p_1 q_1 \quad A^+ = p_1 \quad A = \frac{1}{2} \left( \frac{1 - e^{-2z p_1 q_1}}{z p_1 q_1} \right) q_1.$$

Under this realization,  $H^{(k)}$  is an integrable Hamiltonian with  $k$  degrees of freedom and integrals of the motion  $C^{(l)}$  ( $l = 2, \dots, k$ ).

#### 4.4. Integrable Hamiltonians and the reflection equation algebra

The RE algebra  $\mathcal{A}$  [16] provides a representative example of a four-dimensional comodule algebra with two Casimir elements. The commutation rules of the RE algebra are

$$\begin{aligned} [\alpha, \beta] &= (q - q^{-1})\alpha\gamma & \alpha\gamma &= q^2\gamma\alpha \\ [\alpha, \delta] &= (q - q^{-1})(q\beta + \gamma)\gamma & [\beta, \gamma] &= 0 \\ [\beta, \delta] &= (q - q^{-1})\gamma\delta & \gamma\delta &= q^2\delta\gamma \end{aligned}$$

and the two central elements of  $\mathcal{A}$  are

$$c_1 = \beta - q\gamma \quad c_2 = \alpha\delta - q^2\beta\gamma.$$

The RE algebra is a  $GL_q(2)$ -comodule algebra. We recall the  $GL_q(2)$  commutation rules of a  $GL_q(2)$  element  $T$ ,

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

which read

$$\begin{aligned} ab &= qba & ac &= qca & [a, d] &= (q - q^{-1})bc \\ bd &= qdb & cd &= qdc & [b, c] &= 0. \end{aligned}$$

The Hopf algebra structure in  $GL_q(2)$  is given by the matrix multiplication of two group elements:

$$\Delta(T) = \begin{pmatrix} \Delta(a) & \Delta(b) \\ \Delta(c) & \Delta(d) \end{pmatrix} = T_1 \cdot T_2 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \cdot \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$$

If we rewrite the generators of the RE algebra in matrix form

$$K = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

the (left) coaction map  $\phi^{(2)} : \mathcal{A} \rightarrow GL_q(2) \otimes \mathcal{A}$  will be given by

$$\phi^{(2)}(K) = \begin{pmatrix} \phi^{(2)}(\alpha) & \phi^{(2)}(\beta) \\ \phi^{(2)}(\gamma) & \phi^{(2)}(\delta) \end{pmatrix} = T \cdot K \cdot T^t$$

where  $T^t$  is the transpose of  $T$ . Therefore, the  $k$ th coaction map

$$\phi^{(k)} : \mathcal{A} \rightarrow \overbrace{GL_q(2) \otimes GL_q(2) \otimes \cdots \otimes GL_q(2)}^{k-1} \otimes \mathcal{A}$$

will be given by

$$\phi^{(k)}(K) = \begin{pmatrix} \phi^{(k)}(\alpha) & \phi^{(k)}(\beta) \\ \phi^{(k)}(\gamma) & \phi^{(k)}(\delta) \end{pmatrix} = \left\{ \prod_{l=1}^{k-1} T_{k-l} \right\} \cdot K \cdot \left\{ \prod_{l=1}^{k-1} T_l^t \right\}$$

where

$$T_l = \begin{pmatrix} a_l & b_l \\ c_l & d_l \end{pmatrix} \quad T_l^t = \begin{pmatrix} a_l & c_l \\ b_l & d_l \end{pmatrix}.$$

If we consider as the Hamiltonian any function on  $\mathcal{A}$

$$H = H(\alpha, \beta, \gamma, \delta)$$

the integrability of the Hamiltonian defined as

$$H^{(m)} = \phi^{(m)}(H(\alpha, \beta, \gamma, \delta)) = H(\phi^{(m)}(\alpha), \phi^{(m)}(\beta), \phi^{(m)}(\gamma), \phi^{(m)}(\delta))$$

is guaranteed by the  $2(m-1)$  images of the Casimir functions  $c_1$  and  $c_2$ , that can be written in the following factorized form:

$$c_1^{(k)} = \phi^{(k)}(c_1) = \left\{ \prod_{l=1}^{k-1} \det_q T_l \right\} c_1 \quad 2 \leq k \leq m$$

$$c_2^{(k)} = \phi^{(k)}(c_2) = \left\{ \prod_{l=1}^{k-1} (\det_q T_l)^2 \right\} c_2 \quad 2 \leq k \leq m$$

where the  $q$ -determinant on each  $GL_q(2)$  copy is given by

$$\det_q T_l = a_l d_l - q b_l c_l.$$

It can be easily proved that

$$\begin{aligned} [H^{(m)}, c_1^{(k)}] &= 0 & [H^{(m)}, c_2^{(p)}] &= 0 & 2 \leq k, p \leq m \\ [c_1^{(l)}, c_1^{(n)}] &= 0 & [c_2^{(l)}, c_2^{(n)}] &= 0 & 2 \leq l < n \leq m \\ [c_1^{(k)}, c_2^{(p)}] &= 0 & & & k, p = 2, \dots, m. \end{aligned}$$

In general, coaction-induced integrable systems on quantum homogeneous spaces can be constructed in the same way.

## 5. Concluding remarks

A further generalization of the construction given in theorem 1 is the following. Let us consider a set  $\{A_1, \dots, A_N\}$  of (classical or quantum) Jordan–Lie algebras, then  $A_1 \otimes \dots \otimes A_N$  is again endowed with a Jordan–Lie algebra structure. Let us suppose that we are able to find a set of Jordan–Lie subalgebras  $\{B_1, \dots, B_N\}$  such that

$$B_1 \subset A_1 \otimes A_2 \quad B_{i+1} \subset B_i \otimes A_{i+2} \quad i = 1, \dots, N - 2.$$

Let us also assume that a Casimir element  $C_{B_i} \in B_i$  with respect to the bracket on  $B_i$  exists for any  $i$ . If we denote

$$\begin{aligned} C_1 &= C_{B_1} \otimes \overbrace{1 \otimes \dots \otimes 1}^{N-2} \\ &\vdots \\ C_{N-2} &= C_{B_{N-2}} \otimes 1 \\ C_{N-1} &= C_{B_{N-1}} \end{aligned}$$

the following theorem holds:

**Theorem 2.** *The set  $\{C_1, \dots, C_{N-1}\}$  are mutually commuting elements in  $A_1 \otimes \dots \otimes A_N$  and central elements for  $B_{N-1}$ .*

**Proof.** We have the chain of inclusions:

$$B_i \subset B_{i-1} \otimes A_{i+1} \subset B_{i-2} \otimes A_i \otimes A_{i+1} \subset \dots \subset B_1 \otimes A_3 \otimes \dots \otimes A_{i+1}$$

so that ( $i \leq j$ )

$$\begin{aligned} [C_i, C_j] &\in [C_i \otimes \overbrace{1 \otimes \dots \otimes 1}^{j-i} \otimes \overbrace{1 \otimes \dots \otimes 1}^{N-j}, B_i \otimes A_{i+1} \otimes \dots \otimes A_{j+1} \otimes \overbrace{1 \otimes \dots \otimes 1}^{N-j}] = 0 \\ [C_i, B_{N-1}] &\in [C_i \otimes \overbrace{1 \otimes \dots \otimes 1}^{N-i}, B_i \otimes A_{i+1} \otimes \dots \otimes A_N] = 0. \quad \square \end{aligned}$$

Hence, for an arbitrary element  $H \in B_{N-1}$ , the set  $\{H, C_2, \dots, C_{N-1}\}$  yields an Abelian subalgebra of  $A_1 \otimes \dots \otimes A_N$  of dimension  $N$ .

On the other hand, the freedom to choose different homomorphisms in each step of the ‘duplication’ process should also be mentioned as another possible generalization of the comodule algebra approach. In other words, it would be possible to define the chain of homomorphisms without making use of an iteration of the same elementary map. Work on these lines is in progress.

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